"Sur la gerbe de Kaletha, la courbe et l'ensemble de Kottwitz"

Simons 2022
Edinburgh

Extension du domaine de la lutte

1

Purpose: * Understand the work of Kaletha on rigid inner forms

* Extend the geometric constructions in my joint work with Scholze to reach all inner forms of a reductive group / local field and not just the extended pure inner forms

$E|\mathbb{Q}_p$ - $\bar{E}$ - $\Gamma = Gal(\bar{E}|E)$ - G/E reductive group

Geometric interpretation of Kaletha's gerbe

Kaletha defines a rigid Galois gerbe over E banded by

$\to$ 2-cocycle $\Gamma \times \Gamma \to u(\bar{E})$

no automorphisms

$\hookrightarrow H^1(E, u) = 0$

$u = \varprojlim_{\substack{E'/E \\ n \geq 1}} \mathrm{Res}_{E'/E} [\mu_n]$ pro-diagonaliz

using class field theory: $H^2(E, u) = \widehat{\mathbb{Z}} \ni -1 \leftarrow$ class of Kaletha's gerbe.

uses the cohomology of this gerbe wt. coefficients in G to define a notion of rigid inner form of G that extends Vogan's notion of pure inner form of Vogan given by $H^1(E,G)$.

→ justifies his approach via a global version of this gerbe that is compatible with global transfert factors for endoscopic transfert

* <u>Curve approach</u>: justification via geometry.

- Starting point of the approach: $\eta_X = \eta_{cft}$ — used to define Kaletha's gerbe

the curve

via $\underset{\eta_{cft}}{H^2(E,\mu_n)} \simeq H^2_{et}(X,\mu_n) \ni \eta_X$

- Let $T = \varprojlim_{E'/E} \mathrm{Res}_{E'/E}\, \mathbb{G}_m$ pro-torus.

$$1 \to u \to \underbrace{\tilde{T}}_{\text{universal cover of } T} \to T \to 1$$

$$\begin{array}{c} X = X_E \\ \downarrow \\ \operatorname{Spec} E \end{array}$$ the schematical curve associated to $\hat{E}^\flat$

Canonical point $\infty_E \in X_E$ residue field $\hat{E}$

$$\begin{array}{c} X_{E'} \ni \infty_{E'} \\ \downarrow \\ X_E \ni \infty_E \end{array}$$ for $\bar{E} | \underbrace{E' | E}_{\text{finite degree}}$

$$N_{E'/E}\left(\mathcal{O}_{X_{E'}}(\infty_{E'})\right) = \mathcal{O}_{X_E}(\infty_E)$$

$\rightsquigarrow \left(\mathcal{O}_{X_{E'}}(\infty_{E'})\right)_{E'|E}$ defines a <u>canonical</u> t-torsor

$$\begin{array}{c} \mathbb{T} \\ \downarrow \Big) t \\ X \end{array} \qquad \text{Bu} \left(\begin{array}{c} \mathcal{X} = [\mathbb{T}/\tilde{t}] \\ \downarrow \\ X \end{array} \right. \quad \left[\begin{array}{c}\text{gerbe} \\ \text{of roots of } \mathbb{T}\end{array}\right]$$

$\left[\left[\underline{\text{Th}}\text{: Véa } X \to \operatorname{Spec} E, \quad \mathcal{X} \simeq X \times_{\operatorname{Spec} E} \text{Kal} \right]\right]$

as gerbes banded by u $/X$. unique up to a 2-isomorphism

→ gives a purely geometric definition of $\begin{array}{c} \mathrm{Kal} \\ \downarrow \text{) } B\mu \\ \mathrm{Spec}\, E \end{array}$ via

$$H^2(E,\mu) \xrightarrow{\sim} H^2_{\text{proe}}(X,\mu).$$

<u>To reach all inner forms</u>

$$\breve{E} = \widehat{E^{un}} \circlearrowleft \sigma$$

$$B(G) = G(\breve{E}) / \sigma\text{-conj.}, \quad b \sim g b g^{-\sigma}$$
$$= G\text{-isocrystals}/\sim$$

$[b] \in B(G)$, $G_b = \sigma$-centralizer of b

$B(G)_{bsc} = \{[b] \,/\, \nu_b \text{ central}\}$ "isoclinic" G-isocrystals

$\Rightarrow G_b =$ inner form of G.

$\{G_b \,/\, [b] \text{ basic}\} =: \{\text{extended pure inner forms of } G\}$

Via $\underbrace{H^1(E,G)}_{\text{unit root } G\text{-isocrystals}} \subset B(G)_{bsc}$ This defines a generalization of Vogan's pure inner forms.

* <u>Problem</u>: If Z_G not connected then $B(G)_{bsc} \to B(G_{ad})_{bsc} = H^1(E, G_{ad})$

$$[b] \longmapsto [G_b]$$

is not surjective in general.

Ex: For SL_n the only extended pure inner form is SL_n itself.

Better understood via the curve: recall

$$\begin{tikzcd} & X \arrow[d] \\ (D,\varphi) \mapsto \mathcal{E}(D,\varphi) & \mathrm{Kott} \arrow[d, "B\mathbb{D}"] \\ & \mathrm{Spec}\, E \end{tikzcd}$$

given by $(D,\varphi) \mapsto \mathcal{E}(D,\varphi)$

Kottwitz gerbe of fiber functors on Isocrystals

Induces $\underbrace{H^1_{ét}(\mathrm{Kott}, G)}_{B(G)} \xrightarrow{\simeq} H^1_{ét}(X, G)$

obstruction to surjectivity of $B(G) \to B(G_{ad})$

Exact sequence $H^1_{ét}(X, G) \to H^1_{ét}(X, G_{ad}) \to H^2_{ét}(X, Z_G)$

Recall: $\forall\, T/E$ torus $H^2_{ét}(X, T) = 0$. (in particular $Br(X) = 0$)

$\Rightarrow H^2_{ét}(X, Z_G) \xrightarrow{\simeq} H^2_{ét}(X, \underbrace{\pi_0(Z_G)}_{\text{finite } E\text{-group scheme}})$

Canonical way to kill coho. classes in $H^2_{ét}(X, A)$ — A finite group scheme

$$H^2_{ét}(X,A) = \{\text{Gerbes}/X \text{ banded by } A\}/\sim$$

$\to$ if $c \in H^2_{ét}(X, A)$ is s.t. $c = [\mathcal{G}]$ $\quad \mathcal{G} \xrightarrow{BA} X$

$$H^2_{et}(X, A) \to H^2_{et}(\mathcal{G}, A)$$
$$c \longmapsto 0$$

Following those ideas we prove.

Th: $\mathcal{X} = X \times \mathrm{Kol} \simeq [T/\tilde{E}]$

$$\begin{array}{c} \mathcal{X} \\ \downarrow \\ \mathrm{Kott} \times \mathrm{Kol} \\ \downarrow \\ \mathrm{Spec}\, E \end{array} \Big) \; B(D \times u)$$

We have for any diagonalizable group D/E
$$H^2_{et}(\mathrm{Kott} \times \mathrm{Kol}, D) = H^2_{et}(\mathcal{X}, D) = 0.$$
In particular $H^1_{et}(\underset{\mathcal{X}}{\mathrm{Kott} \times \mathrm{Kol}}, G) \twoheadrightarrow H^1_{et}(\underset{\mathcal{X}}{\mathrm{Kott} \times \mathrm{Kol}}, G_{ad})$

<u>The extended Kottwitz set</u>

Recall: X topos, A abelian group in X

$\mathcal{X}$ gerbe banded by A

G group in X

Exact sequence

$$1 \to H^1(X,G) \to H^1(\mathcal{X},G) \to H^0(X, \mathcal{H}om(A,G)/G)$$

Conjugation

Ex: $H^1_{\text{ét}}(\text{Kott}, G) = B(G)$

$$1 \to H^1(E,G) \to B(G) \to \left[\mathcal{H}om(\mathbb{D}_E, G_E)/G(E)\right]^\Gamma$$
$$[b] \mapsto [\nu_b]$$

Def: $Be(G) = \{ e \in H^1_{\text{ét}}(\text{Kott} \times \text{Kal}, G) \mid \lambda_e : u \to ZG \}$
i.e. is central

Exact sequence:
$$1 \to B(G) \to Be(G) \xrightarrow{\lambda} \mathcal{H}om(u, ZG) \to 1$$
$$b \mapsto \lambda_b$$

Recall Kottwitz theory of $B(G)$: Extension of κ and $\pi_1(G)$:

$\exists$
$$\begin{tikzcd}
1 \arrow[r] & B(G) \arrow[r] \arrow[d, "\kappa"] & Be(G) \arrow[r] \arrow[d, "\kappa"] & \mathcal{H}om(u, ZG) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \pi_1(G)_\Gamma \arrow[r] & \pi_1(G)^e_\Gamma \arrow[r] & \mathcal{H}om(u, ZG) \arrow[r] & 1
\end{tikzcd}$$

Th: (1) $\kappa : Be(G)_{bas} \xrightarrow{\sim} \pi_1(G)^e_\Gamma$

(2) $Be(G) \twoheadrightarrow Be(G_{ad})$ and thus $Be(G)_{bas} \twoheadrightarrow H^1(E, G_{ad})$

$[\Rightarrow \forall G'$ inner form of G $\exists\, b \in Be(G)_{bas}$, $G_b \simeq G']$

<u>Geometrization</u>: Can define

$$\mathrm{Bun}_G^e = \text{Artin } v\text{-stack}$$

$\downarrow$ = stack of G-bundles on "$X \times \mathrm{Kol}$", $\mathcal{E}$,

$\mathrm{Spd}\,\overline{\mathbb{F}}_q$ s.t. $\lambda_{\mathcal{E}}$ is central, $\lambda_{\mathcal{E}} : \ldots \to Z_G$

<u>Th</u>: (1) $|\mathrm{Bun}_G^e| = B_e(G)$

(2) $|\mathrm{Bun}_G^e| \xrightarrow{\kappa} \pi_1(G)_\Gamma^e$ is locally constant

(3) $\mathrm{Bun}_G \underset{\text{open/closed}}{\subset} \mathrm{Bun}_G^e$ is $\kappa^{-1}(\pi_1(G)_\Gamma)$

(4) Thus, $\mathrm{Bun}_G^e = \coprod_{\alpha \in \pi_1(G)_\Gamma^e} \underbrace{\mathrm{Bun}_G^{e,\alpha}}_{\mathrm{Bun}_{G_b} \text{ if } b \in B_e(G)_{bsc},\ \kappa(b) = \alpha}$

<u>Conjecture</u>: $\underbrace{\mathrm{Coh}_{Nilp}\left(\mathrm{LocSys}^e_{\hat{G}/\mathbb{Z}_\ell}\right)}_{\text{extended version of the stack of Langlands parameters}} \simeq \mathrm{Dlis}\left(\mathrm{Bun}_G^e, \mathbb{Z}_\ell\right)$ (ℓ good)

if G quasi-split after fixing a Whittaker datum

$\Rightarrow$ geometrization conjecture for any reductive gp. $/E$, for example inner forms of SL_n !!

- Important from the Langlands point of view. Endoscopy theory of $Sp \neq$ the one of GSp.